All results in Ch3 (on limits and computation rules) can be used except explicitly stated differently (in Q6*).

1. Let $r \in (0,1)$, and (x_n) a contractive sequence with rate r :

$$|x_{n+2} - x_{n+1}| \leq r |x_{n+1} - x_n| \quad \forall\, n \in \mathbb{N}$$

Show that

(i) $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$

(ii) $\sum_{n=1}^{\infty} x_n$ converges

(iii) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$

(iv) $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$ (comparison with $1 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$)

(v) $\sum_{n=1}^{\infty} \frac{1}{n!}$ (comparison with (iii))

2. Let $e_n := \left(1 + \frac{1}{n}\right)^n$, i.e.

$$e_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{n(n-1)\cdot \cdots \cdot (n-k+1)}{k!}\left(\frac{1}{n}\right)^k + \cdots$$

$$+ \cdots + \frac{n(n-1)\cdots(n-(n-2))}{(n-1)!}\cdot\left(\frac{1}{n}\right)^{n-1}$$

$$+ \frac{n(n-1)\cdots(n-(n-1))}{n!}\cdot\left(\frac{1}{n}\right)^n$$

Show that $(e_n) \uparrow$ and bounded and hence $e := \lim_n e_n$ exists in $[1,3]$.

3.* Show that $\lim_n \frac{n^{100}}{(1+\varepsilon)^n} = 0 \quad \forall\, \varepsilon > 0$

4. For each of the cases
   (i) $x_1 = 1$,
   (ii) $x_1 = 10$,

show that $x := \lim_n x_n$ exists in $\mathbb{R}$ (and determine the value of x), where
$$x_{n+1} = 2 + \frac{x_n}{2} \quad \forall\, n \in \mathbb{N}.$$

5.* Let $a > 0$. We know that $\sqrt{a}$ exists in $(0, \infty)$ (can you do this?). Below is a "practical way" to show not only its existence but also a corresponding approximation/numerical procedure. Pick $x_1 > 0$ such that $x_1^2 > a$, and define
$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(i) Show that if $x := \lim_n x_n$ exists in $(0, \infty)$ then $x = \frac{a}{x}$ (and so x is the (positive) sq. root of a). By (ii) & (iii) below, the limit x does exist!

(ii) Show that $x_n^2 \geq a \quad \forall\, n \in \mathbb{N}$
(Hint: $x_{n+1}^2 = \frac{1}{4}\left(x_n^2 + 2a + \left(\frac{a}{x_n}\right)^2\right) \geq a$ because $\left(x_n - \frac{a}{x_n}\right)^2 \geq 0$)

(iii) $(x_n) \downarrow$ (Hint: by (ii), $\frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \leq x_n$).

6.* Suppose $\lim_n x_n = 6$. Show in the $\varepsilon - N$ terminology (and definitions only) that

$$\lim_n \frac{x_n^3 + 4}{x_n - 5} = 220.$$

(properties of $\mathbb{R}$ are allowed).

7. Q1 - Q12 of § 3.3 in Bartle.